REPORT No. 21

THEORY OF AN AIRPLANE ENCOUNTERING GUSTS, II

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INTRODUCTION.

This discussion is an immediate continuation of my previous treatment of the subject published in the First Annual Report of the National Advisory Committee for Aeronautics, Washington, 1915, pages 52–75 (S. Doc. 268, 64th Cong., 1st sess., reference to which will be by pages). The notations of that work will be continued without change except as hereafter noted.

PERIODIC LONGITUDINAL GUSTS.

That there is a certain degree of periodicity in gusts is obvious from casual observation, from the records of scientific observatories like Blue Hill, and from the familiar fact that all such phenomena in nature reveal a general tendency toward periodicity. Needless to say the periodicity is not mathematically exact in its regularity nor indefinite in continuance.

The object, however, of an investigation of the effect of periodic gusts on an airplane can for practical purposes be no other than to reveal any exceptional effects that periodic, as compared with single, gusts may have upon the flight of the machine; and these exceptional effects will probably be indicated with sufficient practical completeness by an analysis built on the assumption of strict periodicity, long continued in operation—the phenomenon most to be feared being resonance.

The longitudinal gusts are in, 1°, head-on velocity u_1; 2°, vertical velocity w_1; 3°, rotary velocity q_1. Very little is known as to the nature of rotary gusts (p. 65) and hence 3° may be left aside. It is not easy to see how vertical gusts can have any pronounced periodicity; the disturbance of the airplane's motion by vertical gusts is (p. 64), except for very sharp gusts, essentially a convection of the machine with and by the gust; for both these reasons 2° may be discarded. This leaves only 1°—periodicity in the head-on gustiness—as likely to be of interest.

[1] Reprinted from Proceedings American Philosophical Society, vol. lvi, 1917.

The gust may be assumed in the form

$$u_1 = J \sin pt \quad \text{or} \quad u_1 = Je^{ipt}. \tag{1}$$

The differential equations are (p. 59)

$$\begin{aligned} f(D)u &= -(0.128D^3 + 1.160D^2 + 3.385D + 0.917)u_1, \\ f(D)w &= -D^2(0.557D + 2.458)u_1, \\ f(D)\theta &= -0.02851Du_1, \\ \text{with } f(D) &= D^4 + 8.49D^3 + 24.5D^2 + 3.385D + 0.917 \\ &= (D^2 + 8.359D + 23.37)\ (D^2 + 0.1308D + 0.03924). \end{aligned} \tag{2}$$

In the previous investigation it was found that the short-period heavily damped oscillation was not of much significance except in the case of a sharp up-gust (pp. 62–69), and that its significance in that case was not revealed in the major motion of the machine but in the initial acceleration (or stress) upon it. It may therefore be expected that for periodic head-on gusts the short-period motion will be negligible in its effects. It is consequently desirable to carry out the numerical analysis in such a way as to separate, so far as may be, the short and long natural periods of the machine.

Let us separate into partial fractions the operator

$$\frac{1}{f(D)} = \frac{1}{(D^2 + 8.359D + 23.37)\ (D^2 + 0.1308D + 0.03924)},$$

or

$$\frac{1}{f(D)} = \frac{0.016D + 0.089}{D^2 + 8.359D + 23.37} + \frac{-0.01601D + 0.24263}{D^2 + 0.1308D + 0.03924}. \tag{3}$$

The first fraction has to do with the short, the second with the long oscillation. The two operators are to be applied to certain expressions derived from (1) by substitution in (2).

If $D = ip$, the numerators of (3) have the respective magnitudes

$$(0.089^2 + 0.016^2p^2)^{1/2} \text{ and } (0.0426^2 + 0.016^2p^2)^{1/2}.$$

For $p = 0$, the second is about half the first; for $p = \infty$, the two are equal; the numerators therefore do not differ greatly in magnitude for any value of p.

The ratio of the denominators is

$$R = \left[\frac{(.03924 - p^2)^2 + .1308^2p^2}{(23.37 - p^2)^2 + 8.329^2p^2} \right]^{1/2},$$

and is very small when p is less than 1. For larger values of p, we have approximately

$$1/R^2 = 1 + 23.2/p^2 + 545/p^4.$$

Hence the short oscillations may be neglected when $p < 1$ without introducing much error; but as p increases beyond the value 1, the importance of the short oscillation grows rapidly.

Consider first the case $p<1$, neglecting the short oscillation. The particular solutions for u, w, and θ, that is, I_u, I_w, I_θ, are obtained from the imaginary parts of

$$
\begin{aligned}
\frac{u}{J}&=\frac{-.01601pi+.04263}{-p^2+.1308pi+.03924}\,(.128p^3i+1.16p^2-3.385pi-.917e^{ipt}.\\
\frac{w}{J}&=\frac{-.01601pi+.04263}{-p^2+.1308pi+.03924}\,(.557p^3i+2.458p^2)e^{ipt},\\
\frac{\theta}{J}&=\frac{-.01601pi+.04263}{-p^2+.1308pi+.03924}\,(-.02851pi)e^{ipt}.
\end{aligned}
\tag{4}
$$

To estimate the value of p corresponding to the maximum disturbance we may examine the amplitude of θ/J, which is

$$
\text{amp.}\frac{\theta}{J}=.02851p\left[\frac{(.04263)^2+(.016p)^2}{(.03924p^2)^2+(.1308p)^2}\right]^{1/2}. \tag{5}
$$

The calculation gives $p^2=0.0394$ or $p=0.1985$. The value of the amplitude is then about $0.0095J$ radians or $0.54J$ degrees. If J should be 20 ft./sec., the forced oscillation would have an amplitude of about 10°.

As the use of $p=0.1985$ in calculating is somewhat more complicated than the use of $p=0.2$, and as the change from 0.1985 to 0.2 does not materially alter the amplitude of the forced oscillation (and probably does not exceed the error of observations), we may use $p=0.2$ in calculating the effect of a periodic gust of maximum resonance on the airplane. We shall first note that for $p=0.2$ the ratio of the amplitudes of the two fractions in (3) is of the order 400 to 1, and the first fraction is therefore entirely negligible in determining the particular integrals.

For the second fraction we have the complex value

$$
\frac{4.263\ -\ .32i}{2.616i\ -\ .076}=\frac{(4.275,\ -\ 4.3°)}{(2.617,\ 91.6°)}=\left(\frac{4.275}{2.617},-95.9°\right),
$$

where the parentheses contain the polar coordinates of the complex numbers. The expressions into which this is multiplied to determine the coefficients of e^{ipt} are for u/J, w/J, θ/J, respectively,

$$
\begin{aligned}
&-0.922-0.676i=(1.144,\ 216.24°),\\
&\ \ 0.0983+0.00456i=(.0984,\ 2.67°),\\
&-0.0057i=(.0057,-90°).
\end{aligned}
$$

Hence the values of u/J, w/J, θ/J are

$$
\begin{aligned}
u/J&=(-.965+1.65i)\ (\cos .2t+i\sin .2t),\\
w/J&=(-.00918-.164i)\ (\cos .2t+i\sin .2t),\\
\theta/J&=(-.00948+.00098i)\ (\cos .2t+i\sin .2t),
\end{aligned}
$$

and

$$
\begin{aligned}
I_u&=J\ (1.65\cos .2t-.965\sin .2t),\\
I_w&=J\ (-.164\cos .2t-.0092\sin .2t).\\
I_\theta&=J\ (.00098\cos .2t-.00948\sin .2t),\\
I_\theta{}'&=J\ (-.0019\cos .2t-.0002\sin .2t),
\end{aligned}
$$

$$
I_{u0}=1.65J,\qquad I_{w0}=-.164J,\qquad I_{\theta 0}=.00098J,\qquad I'_{\theta 0}=-.0019J.
$$

On substituting these values to find the constants of integration (p. 61), it is found that A and C, corresponding to the short oscillation in u, are negligible. Also $B=-1.65J$, $D=.726J$. Hence

$$u=Je^{-.0654t}(-1.65\cos .187t+.726\sin .187t)$$
$$+J(1.65\cos .2t-.965\sin .2t).$$

In like manner (p. 62), A' and C' are small and $B'=.176J$, $D'=-.051J$.

$$w=Je^{-.0654t}(.176\cos .187t-.051\sin .187t)$$
$$-J(.164\cos .2t+.009\sin .2t)-.012Je^{-4.18t}\cos 2.43t.$$

(The last term is added as a check on the initial condition $w=0$.) Finally (p. 62), $A''=.00007J$, $B''=.00104J$, $D''=.0109J$, and

$$\theta=Je^{-.0654t}(-.00104\cos .187t+.0109\sin .187t)$$
$$+J(.00098\cos .2t-.00948\sin .2t)+.00007Je^{-4.18t}\cos 2.43t.$$

Now to find the rise of the machine when the gust strikes it (p. 64).

$$w+115.5\theta=Je^{-.0654t}(.056\cos .187t+1.208\sin .187t)$$
$$-J(.051\cos .2t+1.064\sin .2t).$$

The cosine terms may be omitted. The integration then gives

$$z=5.32J\cos .2t+0.44J-Je^{-.0654t}(2\sin .187t+5.76\cos .187t).$$

A table of values of z may be computed as:

$t=0$,	2,	4,	6,	8,	10,	12,	14,
$z/J=0$,	0,	$-.15$,	$-.54$,	-1.16,	-1.90,	-2.60,	-2.97.

This shows the rise or drop, according as J is negative or positive, during the first quarter minute. The values of z now fall off, pass through 0, and only become large as t nears 35. The natural oscillation is then becoming less effective relative to the forced oscillation which has a double amplitude of about $10.6J$, or 202 ft. if $J=20$ ft./sec.

As the existence of a regular periodic gust for any long time is almost unbelievable, the only real interest in the calculation is in showing that during the first 15 seconds the effect of resonance fails to become so far established that the motion differs appreciably from that due to the simple head-on gust previously studied (p. 74).

In the case of the machine constrained to remain horizontal during flight (by some automatic steering device), the corresponding equations (p. 69) are for $u_1=Je^{ipt}$

$$\frac{u}{J}=-\frac{.128pi+.598}{.598-p^2+4.078pi}e^{ipt},$$
$$\frac{w}{J}=-\frac{.557pi}{.598-p^2+4.078pi}e^{ipt}.$$

As the natural motion is no longer periodic, there can hardly be any such thing as resonance, in the usual acceptation of that term. We can, however, ask what value of p will make w/J a maximum and hence induce the maximum oscillation in the vertical motion. To maximize

$$\frac{p^2}{(.598-p^2)^2+4.078^2p^2} \text{ or } \frac{1}{4.078^2+(p-.598/p)^2}$$

take $p^2=0.598$ or $p=0.774$. The value of w/J is then

$$w/J=-0.136e^{ipt},$$

and the amplitude of w is $0.136J$. The amplitude of the oscillation corresponding to the particular solution I_w is $0.175J$.

Thus again it is seen that the steering device makes the motion far easier than when the machine is free (p. 70). There seems to be no need of carrying out the details of the integration.

NOTE ON RESONANCE.

In defining, by implication, a state of resonance in the calculations above, I have assumed that it was the angle θ which was to be maximized by the proper choice of the frequency p of the applied periodic force. It may be well to take up the theory of resonance in a little greater detail, for there are complications in the kind of system we have here to consider.

A. G. Webster, in his "Dynamics of Particles, etc.," Teubner, 1904, page 175, gives general formulas for resonance and shows that if the damping coefficients are small and if the frequency of the impressed force nearly coincides with that of the natural oscillation, the amplitude of the forced vibration will be relatively large.

This is not enough. For in the first place the damping coefficients in the case of the aeroplane can hardly be regarded as small (they sometimes exceed the frequencies); in the second place, we are not even certain that the motion of the system is wholly oscillatory (some of the roots may be real, and even positive if the machine has a certain amount of dynamical instability); and in the third place, under such conditions, the amplitude of the forced oscillation may be considerably greater when the frequency of the applied force is materially different from that of the system (supposed oscillatory) than when the system and the force are nearly synchronous.

The ordinary theory of simple resonance depends on the equation

$$(D^2+kD+n)x=J \sin pt.$$

The particular solution

$$I_x=\frac{J}{D^2+kD+n} \sin pt$$

is the imaginary part of the expression

$$x=\frac{Je^{ipt}}{n-p^2+kpi}.$$

The amplitude of I_x is the same as the modulus of the complex value x. The modulus of e^{ipt} is 1; that of x is

$$\text{amp. } I_x=\frac{J}{[(n-p^2)^2+k^2p^2]^{1/2}}.$$

To make the denominator a minimum we have merely to minimize

$$(n-q)^2+k^2q,\ \ q=p^2>0.$$

We find $q=n-\frac{1}{2}k^2$, necessitating $n>\frac{1}{2}k^2$. If, then, $n>k^2$, the maximum amplitude of I_x is

$$\text{max. amp. } I_x=\frac{J}{\pm k\ \sqrt{n-\frac{1}{4}k^2}},$$

where the positive or negative sign must be taken according as k is positive or negative. If $n<\frac{1}{2}k^2$, the maximum amplitude for I_x occurs when $p=0$ and is J/n.

The amplitude is large when k or $(n-\frac{1}{4}k^2)$ is small; it is very large when both conditions are satisfied. The largest possible value occurs when $n=\frac{1}{2}k^2$ and is $\sqrt{2}J/k^2$. In this case the applied force has an indefinitely small frequency where the natural oscillation has the frequency $k/\sqrt{2}$. The theory of the system here considered is given by Webster (op. cit., p. 155).

The case which corresponds to that in which we are interested is where the system starts from rest at the position of equilibrium. The motion is then defined by the equation

$$x=\frac{J\sqrt{n-\frac{1}{2}k^2}}{k(n-\frac{1}{4}k^2)}\left(e^{-\frac{1}{2}kt}\cos\sqrt{n-\frac{1}{4}k^2}t-\cos\sqrt{n-\frac{1}{2}k^2}t\right)$$
$$+\frac{J}{2(n-\frac{1}{4}k^2)}\left(\frac{\sqrt{n-\frac{1}{2}k^2}}{\sqrt{n-\frac{1}{4}k^2}}e^{-\frac{1}{2}kt}\sin\sqrt{n-\frac{1}{4}k^2}t-\sin\sqrt{n-\frac{1}{2}k^2}t\right).$$

Under normal conditions this quantity remains tolerably small until the natural motion is nearly damped out or until that motion has time to increase greatly ($k>0$). Even if $n=\frac{1}{4}k^2+\epsilon^2k^2$, the equation becomes

$$x=\frac{2J\epsilon}{k^2}(e^{-\frac{1}{2}kt}\cos\tfrac{1}{2}kt-\cos\epsilon kt)+\frac{2J}{k^2}(\epsilon e^{-\frac{1}{2}kt}\sin\tfrac{1}{2}kt-\sin\epsilon kt),$$

and the conclusion still holds.

For the simple system ordinarily treated for resonance the statement that the motion must be only slightly damped and the frequencies of the natural and forced vibrations must be reasonably near together, is therefore amply justified. The result holds even when $n<\frac{1}{2}k^2$, in which case the maximum amplitude for I_x (resonance) occurs when $p=0$ and is J/n.

The next simplest case is like that which arises in treating the constrained longitudinal motion ($\theta=0$) of the aerpolane (p. 69):

$$\begin{aligned}(D+a)\,u+bw&=-au_1-bw_1,\quad a=.128,\ \ b=-.162,\\ cu+(D+d)w&=-cu_1-dw_1,\quad c=.557,\ \ d=3.95.\end{aligned}$$

The natural motion is given by

$$\Delta'=D^2+(a+d)D+(ad-bc)=0,$$

and in this case by $D^2+4.078D+.598=0$. Here the roots are both real, viz., -3.93 and -0.15. So far as the equation in D is concerned we have the case where k is large and n is small. The equations for the forced motion are

$$\Delta' u = -(aD+n)\,u_1 - bDw_1,$$
$$\Delta' w = -(dD+n)w_1 - cu_1.$$

The question now arises: What is it that is to be a maximum? For some purposes it might be the variables u or w—for example, the whole theory of gusts here given depends on the gust being small and producing small effects, and if by an applied force, the values of u or w should become too large, the theory would become worthless. Again, if the question had to do with the strain on the machine, the derivatives du/dt and dw/dt would be the essential objects of interest, and should be maximized. Finally it might be the values $x=\int u dt$ and $z=\int w dt$—the actual displacements of the machine—which we desired to examine. Let us therefore consider several problems seriatim.

Case 1.—To maximize u with a head-on gust $u_1=e^{ipt}$.

$$u=-\frac{aip+n}{\Delta'}e^{ipt}=-\frac{.128ip+.598}{.598-p^2+4.098ip}e^{ipt}.$$

The maximum value of

$$\frac{.128^2p^2+.598^2}{(.598-p^2)^2+4.098^2p^2}=\frac{.128^2(p^2+21.83)}{p^4+15.59p^2+.3576}$$

occurs when p^2 is o, that is, "resonance" occurs for $p=0$, the amplitude of the force and the oscillation being the same.

Case 2.—To maximize w with a head gust.

This was treated above (sec. 10). The ratio .136 was found; the required value of p was .776.

Case 3.—To maximize u with an up-gust w_1.

$$u=\frac{.162pi}{.598-p^2+4.098pi}e^{ipt}.$$

The condition is $p=.776$ as is case 2; the ratio is .04.

Case 4.—To maximize w with an up-gust.

$$w=-\frac{3.95pi+.598}{.598-p^2+4.098pi}e^{ipt}.$$

The maximum value of

$$\frac{3.95^2p^2+.598^2}{(.598-p^2)^2+4.098^2p^2}=\frac{3.95^2(p^2+.0228)}{p^4+15.59p^2+.3576}$$

occurs when $p^2=.022$ and $p=.15$, and the amplitude ratio is about 1.

We note the very different values of p thus found, namely, 0, 0.15, 0.776, according to the choice of case. If in case 1, we had taken $p=.15$, the amplitude ratio would have been about .7 instead of 1; if $p=.776$ had been assumed, the ratio would have been .37 instead of 1.

Case 5.—If we desired to maximize z we should have had to treat

$$-\frac{1}{i}\,\frac{3.95i+.598/p}{.598-p^2+4.098pi}e^{ipt},$$

which would have given an infinite amplitude ratio for $p=0$.

Now if we turn to the free machine and try to maximize $\int\theta dt$ instead of θ, we have to maximize

$$\frac{.04263^2+.016^2p^2}{(.03924^2-p^2)^2+.1308^2p^2}$$

instead of (5, sec. 6). The value of p^2 is about .0307 and of p about .175 instead of .2 as before. The amplitude ratio is then only slightly in excess (about 4 per cent) of that previously found—in other words the numerical values are such that resonance for θ and for $\int\theta dt$, which is the preponderating term in the expression for z, occurs for considerably different values of p, but the effect is about the same. This may be regarded as validating our procedure (sec. 6) in maximizing θ instead of $\int\theta dt$.

To sum up this discussion of resonance as applied to the aëroplane we may say that the frequencies which produce "resonance" depend largely upon the quantity in which the effect of resonance is to be sought and that the frequency which makes for a strong resonant effect in one quantity may make on another an effect much weaker than the maximum—or it may not.

There remains to discuss the question whether the effect of resonance is practically serious, *i. e.*, whether as in the case of the motion of the machine, above treated, the effect fails to make itself felt until after so long a time that the pilot would be entirely able to deal with it or the wind would really have in all probability ceased to be periodic with the period required.

Now in order to insure that resonance is effective both of itself and as against the natural motion, we should reasonably expect to require, 1°, that the resonant frequency p be large (for if it be small the pilot will have ample time to take care of it), and that, 2°, it be reasonably different from any natural frequency which is only slightly damped (for in the latter case the initial conditions will probably be such as to cause the natural and forced effects to clash for a considerable interval of time).

This problem in its generality is so complicated that I have as yet been unable to determine whether there may be practically serious effects due to resonance, but from the cases I have here treated, from the general considerations which I have advanced, with due regard to the restrictions on p which appear to be reasonable, and from cases which I have examined without mentioning them here, I should judge that resonance is not a practically serious matter in longitudinal motion, and that we may safely confine our attention to gusts of the form $J\,(1-e^{-rt})$.

One type of resonance which deserves consideration is that of the damped harmonic gust $Je^{-nt}\sin pt$. It would be conjectured that if $-n\pm pi$ were nearly equal to a pair of roots of $\Delta=o$, there might arise a considerable disturbance. It is not likely that a gust of this type would exist in reality, but the commencement of any gust might resemble very closely the commencement of such a gust and if the effect of this type were very marked as compared to that of the types already considered, it would be necessary, for the sake of foreseeing the worst that could happen, to discuss this type.

I have not time to take the matter up here. Moreover, I imagine that it would be found that the constants of integration turned out to have such values that the gust, though tuned in damping and in frequency to the natural motion of the machine, did not have very large effects except in cases where n and p were small enough to allow the pilot easily to correct for the disturbance.

The damped periodic gust has been treated by Brodetsky,[1] who finds the amplitude of the particular solution is a maximum (for the machine I am dealing with) when $t=16$ sec. and is then a tolerably large quantity,—but the pilot has a quarter of a minute in which to react to his environment. It is, however, by no means certain that the pilot would have to react so quickly—the constants of integration might turn out, as I have just suggested, such that the motion during the first quarter minute was not far different from that in the case of the simple gust. This was what was found to happen in the case of the periodic gust above treated (sec. 9). The amplitude of the vertical motion so far as the particular solution was concerned turned out to be about $5.3J$, but the constants of integration were such as to postpone the major effect of the particular solution until 30 or 40 seconds had elapsed. If we have a damped harmonic gust and such a postponement were operative, the damping would become effective and the gust might turn out to have at no time an effect much in excess of the maximum effect of a single gust of the form $J(1-e^{-rt})$.

INFINITELY SHARP GUSTS.

In my previous paper I discussed gusts $J(1-e^{-rt})$ rising from zero to J with various degrees of sharpness depending on the value of r—the larger r, the sharper the gust. An infinitely sharp gust would be one for which r was indefinitely large. Such a gust would represent an absolute discontinuity in the velocity of the wind. This is impossible, though it represents a state of aerial motion which may be nearly approached. Moreover, the ifinitely sharp gust could not strike the machine all over at once, and hence the theoretical effect of such a gust on the assumption that the machine is instantaneously immersed must differ from the actual effect upon a machine running into a discontinuity in the wind velocity.

For this reason one may well limit his considerations to finite gusts with a value of r not greater than 5, say, as I did. Nevertheless if the calculation of the effect of an infinitely sharp gust is simpler than for a finite gust and if the limiting motion derived for such a gust is not appreciably different from that for a sharp gust of reasonable sharpness, the discussion of the limiting case will be justified.

Consider first the longitudinal motion and a head-on gust $u^t_1=J$-$(1-e^{-rt})$, r enormously large. According to the symbolic method $D=-r$ must be substituted to find the particular solution for e^{-rt}. As, however, Δ is of the fourth degree in D and all the polynomials upon the right hand are of degree 3 or less, the result of the substitution is easy to find.

[1] Aëronautical Journal, London, 20, 1916, p. 154.

For example, when $u_1 = J(1-e^{-rt})$,

$$I_u/J = -1 - e^{-rt}(.128/r), \qquad I_{u0} = -J,$$
$$I_w/J = -e^{-rt}(.557/r), \qquad I_{w0} = 0,$$
$$I\theta/J = -e^{-rt}(.02851/r^3) = 0, \qquad I\theta_0 = 0,$$
$$I'_\theta/J = e^{-rt}(.02851/r^2) = 0, \qquad I'\theta_0 = 0.$$

The equations of motion are

$$u/J = e^{-4.18t}(.0009 \cos 2.43t + .0032 \sin 2.43t) + e^{-.0654t}(.9991 \cos .187t + .3577 \sin .187t) - 1 - e^{-rt}(.128/r)$$

$$w/J = e^{-4.18t}(.1066 \cos 2.43t - .0435 \sin 2.43t) + e^{-.0654t}(-.1066 \cos .187t + .0352 \sin .187t) - e^{-rt}(.557/r),$$

$$100\theta/J = e^{-4.18t}(-.0402 \cos 2.43t - .0278 \sin 2.43t) + e^{-.0654t}(.0402 \cos .187t - .6683 \sin .187t).$$

The calculation of the constants of integration is much simplified. The terms e^{-rt}/r are retained because the stresses (forces) due to the gust are calculated from du/dt and dw/dt to which these terms make an initial contribution—there is an instantaneous initial stress. When $t = 0$,

$$du/dt = (.128 - .004 - .008 - .085 - .067)J = -.016J,$$

$$dw/dt = (.557 - .446 - .106 + .007 + .006)J = .018J.$$

These are the initial accelerations and should vanish because the gust though infinitely sharp begins at zero. That they do not vanish is due to an accumulation of errors.

Immediately after the initial instant, however, the first terms, viz., .128 and .557, being multiplied by e^{-rt} vanish. The other terms, however, being multiplied by comparatively slow changing functions are not altered. Hence immediately after the first instant there are accelerations $-.128J$ and $-.557J$ along the x and z axes, respectively.

To put it another way, there is a discontinuity in the stress at the initial instant—as might be expected. The amounts of the discontinuities are also just what might be expected, viz, X_uJ and Z_uJ. In like manner for an up-gust the initial discontinuities in acceleration are X_wJ and Z_uJ along the x and z axes. These results could have been foreseen from the differential equations themselves as well as from "common sense." The path in space is not materially different for an infinitely sharp gust from what it is for a reasonably sharp gust.

It may therefore be said that a tolerably good idea of what happens for sharp gusts may be had from the consideration of infinitely sharp gusts.

It has just been stated that the conclusions concerning the initial accelerations may be foreseen from the differential equations. This may be proved as follows: We have

$$\begin{aligned}
(D-X_u)u - X_w w - X_q q - g\theta &= X_u u_1 + X_w w_1 + X_q q_1, \\
-Z_u u + (D-Z_w)w - (Z_q + U)q &= Z_u u_1 + Z_w w_1 + Z_q q_1, \\
-M_u u - M_u w + (k_B{}^2 D - M_q)q &= M_u u_1 + M_w w_1 + M_q q_1, \\
D\theta - q &= 0,
\end{aligned} \qquad (6)$$

where the equations have been reduced to four involving only the first derivatives of the four variables u, w, q, θ, with the initial conditions $u=w=q=\theta=0$, by the device of choosing $q=D\theta$ as an independent variable so as to eliminate the second derivatives.

These equations determine the first derivatives at the initial instant or at any instant in terms of the values of the variables at that instant, namely,

$$
\begin{aligned}
Du&=X_u u+X_w w+X_q q+g\theta+X_u u_1+X_w w_1+X_q q_1,\\
Dw&=Z_u u+Z_w w+Z_q q+Uq+Z_u u_1+Z_w w_1+Z_q q_1,\\
k_B{}^2 Dq&=M_u u+M_w w+M_q q+M_u u_1+M_w w_1+M_q q_1,\\
D\theta&=q.
\end{aligned}
\tag{7}
$$

At the initial instant u, w, q, θ vanish.

With an infinitely sharp gust u_1, w_1, q_1 may be considered as not vanishing, but as starting at finite values, J_u, J_w, J_q. The derivatives are then at the initial instant

$$
\begin{aligned}
Du&=X_u J_u+X_w J_w+X_q J_q,\\
Dw&=Z_u J_u+Z_w J_w+Z_q J_q,\\
k_B{}^2 Dq&=M_u J_u+M_w J_w+M_q J_q,\\
D\theta&=0.
\end{aligned}
\tag{8}
$$

The first two equations give the X and Z accelerations of the machine which determine the stresses as the accelerations times the mass.

We have, for numerical values,

$$
\begin{aligned}
Du&=-.128J_u+.162J_w+0J_q, &&\text{if } X_q=0,\\
Dw&=-.557J_u-3.95J_w+0J_q, &&\text{if } Z_q=0,\\
34Dq&=0J_u+1.74J_w-150J_q, &&\text{if } M_u=0.
\end{aligned}
$$

The last equation determines the couple tending to break the machine, by bending in the x-z-plane, on multiplication by the mass m.

That which I have called an infinitely sharp gust is not an impulsive gust. The impulsive gust is both infinitely sharp and infinitely intense, but endures for only an infinitesimal time. The effect of an impulsive gust is to produce instantaneous changes in u, w, q. Such an impulse, like the impulses of ordinary mechanics, puts an infinite strain on the machine for an infinitesimal time, and the only way to tell whether the machine will stand the strain is to take the yielding of the framework into account—it is a problem in elasticity. For the purpose of calculating the *stresses* produced by gusts on the machine I therefore prefer the sharp gust to the impulsive gust.

For the purpose of treating the *motion* of the machine after the gust strikes it—the gust being now a sudden fierce squall in otherwise still air—we have merely to determine the constants of integration from the initial condition u_0, w_0, q_0, and $\theta=0$, where u_0, w_0, q_0 are the impulsively generated velocities. These equations are (p. 61):

$$
\begin{aligned}
u_0&=A+B,\\
w_0&=-4.04A+34.5C-.1058D+.002587D,\\
0&=-.132A-.0946C+.002478B+.005799D,\\
q_0&=.703A+.205C-.001246D+.000084D.
\end{aligned}
\tag{9}
$$

Analytically the effect of the impulsive gust upon the equations for determining the constants of integration is merely to replace the initial values of the particular solutions I_{u0}, I_{w0}, $I_{\theta 0}$, $I'_{\theta 0}$, obtained on the hypothesis of finite gusts, by the respective values $-u_0$, $-w_0$, 0, $-q_0$. The effect of the disturbance may therefore be calculated at once from my equations (23), (24), (25), (26), as soon as the values u_0, w_0, q_0 have been determined.

In the calculation of u_0, w_0, q_0, the same doubt arises as in the theory of any very sharp gust, namely, the effect of the partial immersion of the machine. Is the effect of a blow traveling along a mechanism the same as that of the blow applied instantaneously at all points of the mechanism? The possibility of a difference between the instantaneous immersion and the immersion distributed in time would arise only if, 1°, the machine had time enough to change its orientation appreciably or, 2°, the acquired velocities were sufficient to change the relative wind and thus affect considerably the impulsive pressure.

Even if we assume that no material difference in effect is to be expected, it is difficult to make the proper assumptions to lead to reasonably satisfactory values for u_0, w_0, q_0 for any actual machine whose characteristics are expressed in terms of the mechanical coefficients m, k_B^2, U, and the aërodynamical coefficients X_u, X_w, X_q, Z_u, Z_w, Z_q, M_u, M_w, M_q. It is by no means certain that for a considerable aërial disturbance the finite instantaneous changes in u, w, q can be calculated from the equations (8) by replacing D by the sign $\triangle$ for the increment and taking J_u, J_w, J_q as the intensities of the impulsive gusts; for the nine coefficients X_u, etc., vary with the intensity of the relative wind.

It is for this reason that I have used finite gusts of various degrees of sharpness instead of impulsive gusts. Moreover, it is not certain but the finite gust represents more nearly actual conditions in the air when flying is at all possible.

An article by Brodetsky, with an introduction by Bryan, has recently reached this country,[1] in which impulsive gusts are considered, relative to Bryan's skeleton aëroplane consisting of a forward main plane and rear tail plane. The discussion is both interesting and important as is everything to which Bryan, the great pioneer in this subject, sets his name, but it does not seem to help me, so far as I have yet been able to examine it, in regard to the effect of an impulsive gust upon a machine whose properties are actually determined in the wind tunnel. I have therefore decided to let stand the brief general considerations above.

THE ACTION OF THE AIR SCREW.

In the work to this point, I have made for the discussion of gusts the same assumption concerning the action of the propeller that Hunsaker, Bairstow, and others have made for discussions of stability, namely, that under varying conditions the motor speeds up or slows down so as to deliver a constant thrust along the x-axis.

It would be equally reasonable, from some points of view more reasonable, to assume that under changing conditions of relative air velocity a motor speeds up or slows down so as to deliver the same

[1] Aeronautical Journal, London, 20, 1916, 139-156.

effective horsepower. We should then have the power P equal to the thrust H (taken positive) multiplied by the velocity $-U$:

$$P=-HU=-(H+dH)(U+u),$$
$$UdH+uH=0,$$
$$dH=-H\frac{u}{U}=-P\frac{u}{U^2}. \tag{10}$$

This is an additional force which is directed along the X-axis if the propeller shaft is horizontal for the velocity of flight $-U$. If in the standard condition the shaft is not horizontal there would be components

$$-P\frac{u}{U^2}\cos\alpha, \quad +P\frac{u}{U^2}\sin\alpha$$

along the x and z axes, α being the angle from the horizontal up to the direction of the shaft. Furthermore if the shaft did not pass through the center of gravity there would be a pitching moment $-Phu/U^2$ if h is the distance of the line of the shaft above the center of gravity.

The equations for the natural longitudinal motion would then be

$$\left(D-X_u+\frac{Pg}{mU^2}\right)u-X_w w-(X_q D+g)\theta=0, \tag{11}$$

the other two equations remaining unchanged, if we assume for simplicity that $\alpha=h=0$. The effect of the varying thrust is to change X_u to X_u-Pg/mU^2. We have the value $X_u=-.128$ for this machine. If the effective propeller horsepower were 87 for $U=-115.5$, the value Pg/mU^2 is

$$\frac{Pg}{mU^2}=\frac{87\times550\times32}{1800\times13350}=.063.$$

The modification of the equations of motion on replacing $X_u=-.128$ by $X_u=-.191$ would make an appreciable, though not very serious change.

The determinant Δ would become

$$34D^4+290.8D^3+850.9D^2+165.1D+31.18$$
$$=34(D^4+8.553D^3+25.03D^2+4.856D+.917)$$

as compared with

$$34(D^4+8.490D^3+24.50D^2+3.385D+.917).$$

The rapidly damped oscillation would, as a first approximation, be

$$-4.276\pm2.596i \text{ instead of } -4.245\pm2.545i.$$

The first approximation for the small root would be

$$-.097\pm.177i \text{ instead of } -.069\pm.181i.$$

The damping would be more pronounced and the oscillation a trifle faster.

It may be concluded that whether the screw is supposed to deliver a constant thrust or a constant power is not very important to the theory either of stability or of gusts. It is not unlikely that the actual behavior of the screw lies within the limits set by these two assumptions or sufficiently near to one of the limits to validate the use of either hypothesis.

The Aëronautical Journal, London, 20, 1916, p. 142, quotes Bairstow and Fage as giving the formula

$$dH = -.011 H dV, \quad V \text{ in miles per hour,}$$

which is

$$dH = -.0073 H dV, \quad V \text{ in feet per second.}$$

With $U = 115.5$ numerically we would have for constant power

$$dH = -.00866 H dV, \quad V \text{ in feet per second;}$$

and, if I understand correctly the use of the signs + and − in the quotation, the results are in as good agreement as could be expected in view of the fact that I have no knowledge of the value of U for which the data quoted are given. (If the motor and screw were exactly designed to give a maximum efficiency at a standard speed U, we could not expect the efficiency to be the same at relative air speeds either higher or lower, and this would slightly influence the result.)

EQUATIONS FOR LATERAL MOTION.

The differential equations for the lateral motion of a machine in a gust may be written as (p. 54):

$$\begin{aligned} dv/dt + g\varphi + Ur - Y_v v - Y_p p - Y_r r &= Y_v v_1 + Y_p p_1 + Y_r r_1, \\ A/m.\, dp/dt - L_v v - L_p p - L_r r &= L_v v_1 + L_p p_1 + L_r r_1, \\ C/m.\, dr/dt - N_v v - N_p p - N_r r &= N_v v_1 + N_p p_1 + N_r r_1, \end{aligned} \tag{12}$$

where the terms involving the small unknown product of inertia E have been neglected and gusts of the type v_1, p_1, r_1 have been allowed.

The gust v_1 corresponds to a side wind. A change in the direction of the wind by a small angle would produce such a gust even in absence of any change in the wind velocity. The gust p_1 is a rotary gust tending to produce a bank; as a disturbance in the air it would correspond to a horizontal roller run into end-on (axially). The gust r_1 corresponds to a column of air rotating about a vertical line.

This last is a common type of aerial disturbance, easily observed on a warm day, often of very small diameter compared with the spread of the wings of an aeroplane, and accompanied by a strong rising current of air. Such a vertical vortex, if small, might strike one wing of the machine alone, and, due to the rising current, heel it over suddenly. It is, however, not this small local disturbance which we can consider by our methods here, but the larger and more gentle rotation in the air which might immerse the whole machine many times over and which produces a yawing motion in the machine rather than (primarily) a roll or bank.

31. Place $D = d/dt$. Then the equations are

$$\begin{aligned} (D - Y_v)v + (g - Y_p D)\phi + (U - Y_r)r &= Y_v v_1 + Y_p p_1 + Y_r r_1, \\ -L_v v + (k_A^2 D - L_p)L\phi - L_r r &= L_v v_1 + L_p p_1 + L_r r_1, \\ -N_v v - N_p D\phi + (k_C^2 D - N_r)r &= N_v v_1 + N_p p_1 + N_r r_1, \end{aligned} \tag{13}$$

where $k_A^2 = A/m$ and $k_C^2 = C/m$. The determinant whose vanishing determines the natural motion is

$$\Delta = \begin{vmatrix} D - Y_v & g - Y_p D & U - Y_r \\ -L_v & k_A^2 D^2 - L_p D & -L_r \\ -N_v & -N_p D & k_C^2 D - N_r \end{vmatrix}.$$

Let the cofactors of Δ be

$$\delta_{11} = \begin{vmatrix} k_A^2 D^2 - L_p D & -L_r \\ -N_p D & k_C^2 D - N_r \end{vmatrix} = 2592 D^3 + 23140 D^2 + 8478 D.$$

$$\delta_{12} = \begin{vmatrix} -L_r & -L_v \\ k_C^2 D - N_r & -N_v \end{vmatrix} = -59.55 D - 26.55,$$

$$\delta_{13} = \begin{vmatrix} -L_v & k_A^2 D^2 - L_p D \\ -N_v & -N_p D \end{vmatrix} = -32.84 D^2 - 280.7 D,$$

$$\delta_{21} = \begin{vmatrix} -N_p D & k_C^2 D - N_r \\ g - Y_p D & U - Y_r \end{vmatrix} = -2270 D - 868.8,$$

$$\delta_{22} = \begin{vmatrix} D - Y_v & U - Y_r \\ -N_v & k_C^2 D - N_r \end{vmatrix} = 70.6 D^2 + 44.5 D + 109.9,$$

$$\delta_{23} = \begin{vmatrix} -N_v & -N_p D \\ D - Y_v & g - Y_p D \end{vmatrix} = 28.76,$$

$$\delta_{31} = \begin{vmatrix} g - Y_p D & U - Y_r \\ k_A^2 D^2 - L_p D & -L_r \end{vmatrix} = 4243 D^2 + 36270 D - 1776,$$

$$\delta_{32} = \begin{vmatrix} U - Y_r & D - Y_v \\ -L_r & -L_r \end{vmatrix} = 55.2 D + 111.2,$$

$$\delta_{33} = \begin{vmatrix} D - Y_v & g - Y_p D \\ -L_v & k_A^2 D^2 - L_p D \end{vmatrix} = 36.7 D^3 + 323.1 D^2 + 77.88 D + 27.15,$$

where the numerical values are those arising from the data determined for the Curtiss Tractor (which is the machine under investigation) by Dr. J. C. Hunsaker as given on page 78 of his paper, "Dynamical Stability of Aeroplanes," Smithsonian Misc. Collect., Washington, Vol. 62, No. 5, pp. 1–78, 1916, namely,

$$Y_v = -0.248, \quad Y_p = 0, \quad Y_r = 0,$$
$$L_v = +0.844, \quad L_p = -314, \quad L_r = +55.2,$$
$$N_v = -0.894, \quad N_p = 0, \quad N_r = -27.0,$$
$$k_A^2 = 36.7+, \quad k_C^2 = 70.6-, \quad U = -115.5, \quad g = 32.17.$$

The value of Δ is then $(D - Y_v)\delta_{11} + g\delta_{12} + U\delta_{13}$ or

$$\Delta = 2592 D^4 + 23780 D^3 + 18000 D^2 + 34610 D - 854.$$

This result checks with Hunsaker's (loc. cit., p. 78) as well as probable. The equation $\Delta = 0$ may be written as

$$D^4 + 9.172 D^3 + 6.943 D^2 + 13.35 D - 0.3295 = 0.$$

32. From the last two terms, one root is indicated as $D = 0.02468$; and the correction can readily be found, giving

$$D = 0.02436.$$

There is another root near $D = -8.5$, the exact value being

$$D = -8.542.$$

The other factor of the biquadratic equation is

$$D^2+0.6537D+1.583=0,$$

of which the roots are

$$D=-0.3268\pm 1.215i.$$

The complementary functions for v, ϕ, and r are therefore of the form

$$v=C_{11}e^{.02436t}+C_{12}e^{-8.542t}+e^{-.3268t}(C_{13}\cos 1.215t+C_{14}\sin 1.215t),$$
$$\phi=C_{21}e^{.02436t}+C_{22}e^{-8.542t}+e^{-.3268t}(C_{23}\cos 1.215t+C_{24}\sin 1.215t),$$
$$r=C_{31}e^{.02436t}+C_{32}e^{-8.542t}+e^{-.3268t}(C_{33}\cos 1.215t+C_{34}\sin 1.215t).$$

The particular integrals for any gust may be represented as I_v, I_ϕ, I_r, and their initial values as I_{v0}, $I_{\phi0}$, I_{r0}, the derivative of I_ϕ being I'_ϕ with the corresponding initial values $I'_{\phi0}$.

If, as before (p. 59), we restrict the possible gusts to those of which the functional form is different from any of the four functions entering into the complementary functions, the particular solutions must, on substitution, annihilate the right-hand members of the differential equations, and the relations between the constants C_{ij} of integration may be determined from the two equations

$$(D+0.248)v+32.17\phi-115.5r=0,$$

$$0.894v+0\phi+(70.6D+27.0)r=0.$$

Hence,

$$.2724C_{11}+32.17C_{21}-115.5C_{31}=0,$$

$$.894C_{11}+0C_{21}+28.72C_{31}=0,$$

and

$$C_{11}=-8.326C_{21},\quad C_{31}=.2591C_{21}.$$

Further,

$$-8.294C_{12}+32.17C_{22}-115.5C_{32}=0,$$

$$.894C_{12}+0C_{22}-575.8C_{32}=0,$$

and

$$C_{12}=3.797C_{22},\quad C_{32}=.005897C_{22}.$$

Finally,

$$-.0788C_{13}+1.215C_{14}+32.17C_{23}-115.5C_{33}=0,$$

$$-1.215C_{13}-.0788C_{14}+32.17C_{24}-115.5C_{34}=0,$$

$$.894C_{13}+3.92C_{33}+85.74C_{34}=0,$$

$$.894C_{14}-85.74C_{33}+3.92C_{34}=0,$$

and

$$C_{13}=1041C_{23}+564.8C_{24},\qquad C_{33}=-6.371C_{23}+10.56C_{24},$$

$$C_{14}=-564.8C_{23}+1041C_{24},\qquad C_{34}=-10.56C_{23}-6.371C_{24}.$$

The solutions therefore, so far as concerns the complementary function, are

$$\phi = C_{21}e^{.02436t} + C_{22}e^{-8.542t} + e^{-.3268t}(C_{23}\cos 1.215t + C_{24}\sin 1.215t),$$

$$v = -8.326C_{21}e^{.02436t} + 3.797C_{22}e^{-8.543t} + e^{-.3268t}[(1041C_{23} + 564.8C_{24})\cos 1.215t + (-564.8C_{23} + 1041C_{24})\sin 1.215t],$$

$$r = 0.2571C_{21}e^{.02436t} + 0.005897C_{22}e^{-8.543t} + e^{-.3268t}[(-6.371C_{23} + 10.56C_{24})\cos 1.215t - (10.56C_{23} + 6.371)\sin 1.215t].$$

These equations determine the relative magnitudes of the various sorts of natural motion.

The first term is the slowly amplifying divergence, this machine being slightly unstable laterally. If a side gust is such as to induce a lateral velocity of $-8.326C_{21}$, it induces a bank of C_{21}, an eighth as much in radians or seven times as much in degrees. It is therefore clear that only very small values of C_{21} are admissible for safety. The second term, corresponding to the rapidly damped motion, shows such rapid damping that it can hardly be of importance, except for possible strains on the mechanism, unless C_{22} is so large that the whole work is inapplicable because of the failure of the motions to be small.

The trigonometric terms show that the oscillation in v will be of great amplitude compared with that in ϕ, the factor being about 1200 when ϕ is in radians or 20 when ϕ is in degrees; even the oscillation in r will be over 12 times as great as in ϕ. In other words, the machine may have a large oscillatory side-slip or angular velocity of yaw without much bank, but for the divergent motion the bank is a serious matter for even moderate side-slip.

The initial conditions $\phi = p = v = r = 0$ give

$$\begin{aligned}
0 &= C_{31} + C_{22} + C_{23} + I_{\phi 0},\\
0 &= .02436C_{21} - 8.542C_{22} - .3268C_{23} + 1.215C_{24} + I'_{\phi 0},\\
0 &= -8.326C_{21} + 3.797C_{22} + 1041C_{23} + 564.8C_{24} + I_{v0},\\
0 &= .2571C_{21} + .005897C_{22} - 6.371C_{23} + 10.56C_{24} + I_{r0}.
\end{aligned}$$

These equations must be solved for the four constants C.

$$\begin{aligned}
C_{21} &= -.9839I_{\phi 0} - .1148I'_{\phi 0} + .000740I_{v0} - .02797I_{r0},\\
C_{22} &= -.000149I_{\phi 0} + .1170I'_{\phi 0} - .0000342I_{v0} - .01163I_{r0},\\
C_{23} &= -.01595I_{\phi 0} - .002153I'_{\phi 0} - .000706I_{v0} + .0396I_{r0},\\
C_{24} &= .01468I_{\phi 0} + .001466I'_{\phi 0} - .0004537I_{v0} - .07201I_{r0}.
\end{aligned}$$

The equations from which the particular solutions are obtained are (since $Y_p = N_p = Y_r = 0$):

$$\begin{aligned}
\Delta v &= (D\delta_{11} - \Delta)v_1 + L_p\delta_{21}p_1 + (L_r\delta_{21} + N_r\delta_{31})r_1,\\
\Delta\phi &= D\delta_{12}v_1 + L_p\delta_{22}p_1 + (L_r\delta_{22} + N_r\delta_{32})r_1,\\
\Delta r &= D\delta_{13}v_1 + L_p\delta_{23}p_1 + (L_r\delta_{23} + N_r\delta_{33})r_1,
\end{aligned} \qquad (14)$$

or

$$
\begin{aligned}
\Delta v &= (-640D^3-9522D^2-34610D+854)v_1+(7134D\\
&\qquad\qquad +2732)p_1-(112560D^2+1104700D)r_1,\\
\Delta\phi &= (-59.55D-26.55)Dv_1+(-22150D^2-13970D\\
&\qquad\qquad -34510)p_1+(3895D^2+970D+3062)r_1\\
\Delta r &= (-32.84D-280.7)D^2v_1+(-9030)p_1\\
&\qquad\qquad +(-992D^3-8724D^2-2103D+854)r_1,
\end{aligned}
$$

with

$$\Delta=2592D^4+23780D^3+18000D^2+34610D-854.$$

Motion in Lateral Gusts.

We shall take as before the type $J(1-e^{-rt})$ for that of a single gust.

Case 1.—Side-gust—sharp. $v_1=J(1-e^{-5t})$.

$$
\begin{aligned}
I_v &= J(-1+.01473e^{-5t}), & I_{v0} &= -.98527J,\\
I_\phi &= J(-.001028)e^{-5t}, & I_{\phi 0} &= -.001028J,\\
I'_\phi &= J(.00514)e^{-5t}, & I'_{\phi 0} &= .00514J,\\
I_r &= J(.002706)e^{-5t}, & I_{r0} &= .002706J,
\end{aligned}
$$

$C^{21}=-.000384J,\quad C^{22}=.0005364J,\quad C^{23}=.000809J,\quad C^{24}=.0002445J.$

The equations of motion are

$$
\begin{aligned}
1000\phi/J &= -.384e^{.02436t}+.536e^{-8.542t}-1.028e^{-5t}\\
&\qquad +e^{-.3263t}(.809\cos 1.215t+.2445\sin 1.215t).
\end{aligned}
$$

This is all negligibly small. For the same reason certain terms may be neglected in v and r.

$$
\begin{aligned}
v/J &= .003e^{.02436t}+.002e^{-8.542t}-1+.01473e^{-5t}\\
&\qquad +e^{-.3263t}(.98\cos 1.215t-.2022\sin 1.215t),\\
100r/J &= -.01e^{.02436t}+.271e^{-5t}-e^{-.3263t}(.257\cos 1.215t\\
&\qquad +1.009\sin 1.215t).
\end{aligned}
$$

The effect of the sharp side-gust is to carry the machine sideways with it, but not very powerfully at first—much of the air blows through the machine—the dominating term at first being

$$v=-.2Je^{-.3263t}\sin 1.215t;$$

after a few seconds the dominating term is $v=-J$, with the very slowly growing divergent term effective only after a considerable time. There is a slight yawing oscillation, but the extreme angle of yaw is only about $0.01J$ radians or $J/2$ degrees—the angle being computed as

$$
\begin{aligned}
100\psi/J &= \int_0^t 100r/J.dt=.4(1-e^{.02436t})+.054(1-e^{-5t})-.8316\\
&\qquad +e^{-.3263t}(.8316\cos 1.215t+.0122\sin 1.215t).
\end{aligned}
$$

The actual sidewise velocity is compounded of v and the amount -115.5ψ due to the yaw. Hence

$$y=\int_0^t (v-115.5\psi)dt.$$

For this calculation v and ψ may be simplified to

$$v=-J+Je^{-.3268t}(\cos 1.215t-.2\sin 1.215t),$$
$$100\psi=-.378J-.4Je^{.02346t}+Je^{-.3268t}(.832\cos 1.215t);$$

and

$$y=-.56Jt+18.5J(e^{.02436t}-1)-.146J$$
$$+Je^{-.3268t}(.146\cos 1.215t+.066\sin 1.215t).$$

From this it will be seen that the oscillatory motion is, so far as concerns the lateral displacement, of very small amplitude. The first two terms which are progressive, are the ones which count. Moreover, the displacement is of the same sign as J although the side-slip v is of the opposite sign. This apparent contradiction is due to the change in orientation ψ— the machine moves away from the gust owing to the lateral excess wind pressure, but turns into the gust owing to the moment of the pressures, and by virtue of the great forward velocity, this turning more than makes up, in the displacement, for the side-slipping.

Case 2.—Side gust—mild. $v_1=J(1-e^{-.2t})$.

$$I_v=J(-1+1.0205e^{-.2t}),\qquad I_{v0}=.0205J,$$
$$I_\phi=J(.0004043e^{-.2t}),\qquad I_{\phi 0}=.0004043J,$$
$$I'_\phi=J(-.0000809e^{-.2t}),\qquad I'_{\phi 0}=-.0000809J,$$
$$I_r=J(-.001514e^{-.2t}),\qquad I_{r0}=-.001514J,$$
$$C_{21}=-.000331J,\qquad C_{22}=.00000738J,\qquad C_{23}=-.0000807J,$$
$$C_{24}=.0001055J.$$

It is again seen that there is practically no rolling motion produced by the side gust. For v and r,

$$v/J=.0027e^{.02436t}-1+1.0205e^{-.2t}$$
$$+e^{-.3268t}(-.0244\cos 1.215t+.1554\sin 1.215t),$$
$$100r/J=-.0085e^{.02436t}-.1514e^{-.2t}$$
$$+e^{-.3268t}(.1628\cos 1.215t+.0672\sin 1.215t).$$

(The check $v=0$, $r=0$, when $t=0$, shows that the accuracy has been reduced so that the third place is not sure.) The effects of the gust are qualitatively as before. The oscillatory motion is not pronounced; the ultimate side-slip velocity is $-J$; the ultimate displacement has the same sign as J because the divergent term in $v-115.5\psi$ is positive.

Case 3.—Side gust—oscillatory. When one examines the records made or making at such an observatory as Blue Hill for gustiness in the air, no phenomenon is perhaps more striking than the reasonably periodic side switching of a reasonably steady wind. A south wind, for example, may whip back and forth between southsoutheast and southsouthwest for hours at a stretch, as Prof. Alexander McAdie has been kind enough to show me on some of his records. In the absence of rotary motion, concerning which I am unable to find satisfactory data, the simplest way to figure this change in direction is as a periodic side gust. A machine going south in such a wind would experience an alternating side gust. (The oscillations in the head-on velocity of the wind would be relatively very small except for actual changes in head-on velocity superimposed upon the changes in direction.) It is therefore especially interesting to discuss a periodic side gust—this being the only periodic gust of which we can reasonably be said to know anything at all definite.

Let $v = Je^{ipt}$. We may assume, from our work above that the rolling motion will be small and that the side-slip velocity v will not be of as much importance in determining the path as the angle ψ coupled with the large forward velocity. The complex value of r is

$$r = \frac{(280.7 - 32.84pi)p^2Je^{ipt}}{2592p^4 - 18000p^2 - 854 + i(34610p - 23780p^3)}$$

If at any one place the period of the complete oscillation is $2\pi/n$ with the wind velocity V, the distance traveled by the wind during the time of an oscillation in direction is $2\pi V/n$, and this is the distance between the nodes of the motion. The time required for this machine ($U = -115.5$) to pass over the distance $2\pi V/n$ is $2\pi V/115.5n$. The periodicity of the gust as it appears to the operator of the machine will therefore correspond to the value $p = 115.5n/V$. For instance, if $V = 20$ and the time of an oscillation at one spot were 10 secs. so that $n = 0.63$, the value of p would be about $p = 3.6$, and the oscillations would appear to the pilot as taking place about every $1\frac{3}{4}$ seconds. A slower oscillation, i. e., a longer periodic time, would diminish n and p, $-$an oscillation at one spot every half minute corresponds to a value $p = 1.2$ on the basis of the assumptions made above.

In considering the values of p which make the amplitude of r large, the only hope is to make the term $34610p - 23780p^3$ tolerably small. This means p^2 must be about 1.5. For this value, the modulus of r is about $.03J$ and the modulus of the yawing oscillation corresponding will be about $.025J$. If a wind of 20 foot-seconds is whipping through an angle of 45°, the side gust will be only of about 7 foot-seconds semi-amplitude and the angle of yaw will be in the neighborhood of 0.175 radians or 10°. There is nothing to indicate that this would be fatal, though it would surely be a nuisance.

Owing to the fact that the coefficients of i in both numerator and denominator are relatively small, the angular velocity I_r would be about in phase with the gust v_1, and hence the angle I_ψ would be about quartered in phase. If there were periodically an angle of 10° or 12° between the direction of flight and the relative wind, we should find that we were getting into a region where considerable rolling and pitching might be induced—for as Hunsaker has shown (loc. cit., p. 62) the lateral and longitudinal motions are not strictly independent; but as the machine makes the major part of the relative wind, the directions of flight and of the relative wind never differ greatly—only some 3° at most in the case under consideration.

It seems hardly necessary at this time to go into the calculation of the actual motion; enough has perhaps been accomplished in showing that the oscillation of the direction of the wind induces at most a moderate yawing of the machine. The semiamplitude of 115.5ψ would be, if $J = 7$ foot-seconds, about 20 feet; the center of gravity of the machine would sway back and forth across the line of flight with a total amplitude of 40 feet, until the divergent term became effective.

Case 4.—Rolling gust. $p_1 = J(1-e^{-rt})$. If there were no interaction between v, p, r, the effect on rolling of a rolling gust would be figured from the equation

$$36.7Dp+314p=-314J(1-e^{-rt}),$$

$$p/J=-8.055e^{-8.055t}\int_0^t e^{8.055t}(1-e^{-rt})dt,$$

$$p/J=-1+e^{-8.055t}+\frac{8.055}{8.055-r}e^{-rt}-\frac{8.055}{8.055-r}e^{-8.055t}.$$

This means that for any ordinary sharp gust p rapidly acquires the value $-J$, and ϕ the value $-Jt$ (radians). It must therefore be expected that unless J is very small indeed, the motion will be much disturbed. There will be developed a component of the weight, inducing side slipping, and yawing will rapidly develop—the machine apparently goes off on a spiral dive.

We may make the calculations in detail when $r=1$. Here

$$I_v/J=-3.14-.114e^{-t}, \qquad I_{v0}=-3.25J,$$
$$I_\phi/J=40.4-1.1e^{-t}, \qquad I_{\phi 0}=39.3J,$$
$$I'^\phi/J=1.1e^{-t}, \qquad I'_{\phi 0}=1.1J,$$
$$I_r/J=10.58-.234e^{-t}, \qquad I_{r0}=10.35J.$$
$$C_{21}=-39.1J, \quad C_{22}=.0027J, \quad C_{23}=-.219J, \quad C_{24}=-.163J.$$

The equations of motion become

$$\phi/J=-39.1e^{.02436t}+.003e^{-8.542t}+40.4-1.1e^{-t}$$
$$+e^{-.3268t}(-.219\cos 1.215t-.163\sin 1.215t),$$
$$v/J=324e^{.2436t}+.0103e^{-8.542t}-3.14-.114e^{-t}$$
$$+e^{-.3268t}(-320\cos 1.215t+49.2\sin 1.215t),$$
$$r/J=-10e^{.02436t}+10.58-.234e^{-t}$$
$$+e^{-.3268t}(-.33\cos 1.215t+3.35\sin 1.215t).$$

In the equation for Φ the effective terms are

$$\phi/J=-39(e^{.02436t}-1)=-t(\text{nearly}),$$

and there is a steady divergence in ϕ to the approximate amount $-Jt$ as foreseen. The side-ways velocity v develops more slowly, perhaps, but after one second amounts to something like $300J$. It is clear that J must be very small or the motion becomes disastrous.

It would be of especial interest to know what sorts of magnitudes for J are likely to arise in flight under normal conditions. In so far as experience shows that machines are not liable to roll and side-slip, it is pretty good evidence that aerial rotary motion with axis parallel to the earth is rare and small.

Case 5.—Yawning gust. $r_1=J(1-e^{-t})$.

$$I_v/J=25.67e^{-t}, \qquad I_{v0}=25.67J,$$
$$I_\phi/J=-.0792+.154e^{-t}, \qquad I_{\phi 0}=.075J,$$
$$I'_\phi/J=-.154e^{-t}, \qquad I'_{\phi 0}=-.154J,$$
$$I_r/J=-1-.1235e^{-t}, \qquad I_{r0}=-.1.12J,$$
$$C_{21}=-.006J, \quad C_{22}=-.006J, \quad C_{23}=-.0634J, \quad C_{24}=.0702J.$$

In this case the motion is

$$
\begin{aligned}
\phi/J &= -.006e^{.02436t} - .006e^{-8.542t} - .0792 + .154e^{-t} \\
&\qquad + e^{-.3268t}(-.0634 \cos 1.215t + .0701 \sin 1.215t), \\
v/J &= +.05e^{.02436t} - .023e^{-8.542t} + 25.67e^{-t} \\
&\qquad + e^{-.3268t}(-26.35 \cos 1.215^{t} + 108.9 \sin 1.215t), \\
r/J &= -1 - .1235e^{-t} + e^{-.3268t}(1.145 \cos 1.215t + .222 \sin 1.215t).
\end{aligned}
$$

For moderate values of J, there is nothing serious indicated. The coefficients of the divergent terms are small. There can not be much roll. The most noteworthy phenomenon is the large amount of side slip which is fairly rapidly damped out.

This leaves the rolling gust as the only dangerous type of lateral gust.

The infinitely sharp side gust would produce an initial acceleration Y_vJ.

CONSTRAINED AIRPLANES.

Suppose now that by some automatic steering device the aeroplane were constrained to remain pointing in the same direction, i. e., so that $r=0$ identically. The equations of motion become

$$
\begin{aligned}
(D-Y_v)v + (g - Y_pD)\phi &= Y_vv_1 + Y_pp_1 + Y_rr_1, \\
-L_vv + (k_A{}^2D - L_p)D\phi &= L_vv_1 + L_pp_1 + L_rr_1, \\
-N_vv - N_pD\phi &= N_vv_1 + N_pp_1 + N_rr_1 + F,
\end{aligned} \tag{15}
$$

where Fm is the moment necessary to maintain the constraint. The last equation may be regarded as determining F.

The natural motion of the constrained machine is found from the determinant

$$\Delta' = \delta_{33} = 36.7D^3 + 323.1D^2 + 77.88D + 27.15 = 0.$$

This is a cubic equation which has no positive root.

The negative root is -8.54. The quadratic factor remaining after division by $D+8.54$ is

$$36.7D^2 + 8.746D + 3.18 = 0,$$

of which the roots are

$$D = -0.119 \pm 0.269i.$$

The real part is negative and hence the motion is dynamically stable.

The introduction of the automatic device has removed the instability in the lateral motion. As compared with the complex roots in the free motion, these roots indicate a much slower period and a considerably smaller damping.

On the other hand suppose that the constraint had been such as to keep the machine level, i. e., $\phi=0$ identically. The equations would have been

$$
\begin{aligned}
(D-Y_v)v + (U - Y_r)r &= Y_vv_1 + Y_pp_1 + Y_rr_1, \\
-L_vv - L_rr &= L_vv_1 + L_pp_1 + L_rr_1 + F, \\
-N_vv + (k_C{}^2D - N_r)r &= N_vv_1 + N_pp_1 + N_rr_1.
\end{aligned} \tag{16}
$$

The natural motion would have been determined by

$$\Delta''=\delta_{22}=70.6D^2+44.5D+109.9=0.$$

The roots are

$$D=-.0.315\pm 0.237i.$$

The machine is again stable.

It follows that at high speed this Curtiss tractor, which is laterally unstable when free, becomes quite stable when constrained either to remain on its course or to fly on even keel.

If stabilizers against rolling and turning were provided, the motion would reduce to

$$(D-Y_v)v=Y_v v_1+Y_p p_1+Y_r r_1, \tag{17}$$

and would be stable, $D=Y_v=-0.248$.

It would be a relatively easy matter to discuss the effect of gusts of various types on the airplane constrained in various ways; two equations are much easier to handle than three. Until some definite problem is proposed as important, until some particular constraining device is indicated as likely to be adopted, it may be as well not to go into the calculations, which are quite straightforward.

That a constraint against rolling might be worth while, and would indeed be very valuable if rolling gusts were a common thing, is suggested by the work done on the free machine (sec. 42) where gustiness was seen not to be very serious except for the rolling gust.

DISCUSSION OF METHOD.

I pointed out in my earlier paper that there were several outs about my method of treating gusts. First the gusts must be small. If they are not tolerably small, flying would be too difficult—so that assumption is not wholly unjustifiable. Second, the calculations for determining the individual equations of motion and for determining formulas for the constants of integration are very tedious. Third, the numbers are of such various magnitudes that the arithmetical operations which must be carried out cut down the accuracy of the work a good deal and indeed, unless great care is taken, will lead to illusory or incorrect results. This does not appear to be due to any very rapid variation of the true results calculated from varying data but to the mode of computing.

To offset these inconveniences we have the satisfactory result that once the preliminary calculations are made, many and varied types of gusts may easily be treated, and the further valuable result that the actual motion for each case is known so that not only the initial motion is determined but the whole extent of the motion. This last is necessary for any just appreciation of the effects of periodic gusts and resonance, as has been shown.

For another method of treating gusts reference may be made to a recent paper by Brodetsky and Bryan, "The longitudinal initial motion and forced oscillations of a disturbed airplane," Aeronautical Journal, London, 20, 1916, pages 139–156, which has already been cited in the text.

Much may be said for their method of expansion in series—for some problems the work is decidedly simpler than with my method. It has been my experience, however, that the application of series to the motion of any airplane has its own difficulties and complicated calculations when the motion is to be followed for any reasonable length of time and especially if the machine is defined, as I have always preferred to regard it as defined, by the actual coefficients determined by wind tunnel experiments rather than as Bryan's skeleton plane consisting of a main front plane plus tail plane—even though the results obtained from such a skeleton may be extended to more complicated machines by Bryan's invariant method. (See his Stability in Aviation, Chap. VI.)

The question therefore arises whether there may not be some way of abridging the calculations leading to the actual motion of the machine. Since finishing my work above, I have received the Proceedings of the London Mathematical Society, volume 15, 1917, part 6, in which there is an article on "Normal coordinates in dynamical systems," by T. J. I'A. Bromwich, in which he develops a method of treating the motions of dynamical systems by means of the theory of functions of a complex variable. I wish, in closing, to describe the application of Bromwich's work to the problem in hand.

We have to solve for the longitudinal motion equations of the type

$$\begin{aligned}(D-X_u)u-X_w w-(X_q D+g)\theta&=P_1 e^{\mu t},\\ Z_u u+(D-Z_w)w-(Z_q+U)D\theta&=P_2 e^{\mu t}\\ -M_u u-M_w w+(k_B^2 D^2-M_q D)\theta&=P_3 e^{\mu t},\end{aligned} \tag{18}$$

where u is a real or complex number, the values we have used being $o, -r, \pm pi$. We substutite

$$\begin{aligned}u&=\frac{1}{2\pi i}\int_o e^{\lambda t}\xi d\lambda,\\ w&=\frac{1}{2\pi i}\int_o e^{\lambda t}\eta d\lambda,\\ \theta&=\frac{1}{2\pi i}\int_o e^{\lambda t}\zeta d\lambda,\end{aligned} \tag{19}$$

where the integrals are loop integrals in the complex plane and ξ, η, ζ are any functions of λ. The results are

$$\begin{aligned}&\frac{1}{2\pi i}\int_o[(\lambda-X_u)\xi-X_w\eta-(X_q\lambda+g)\zeta]e^{\lambda t}d\lambda=P_1 e^{\mu t},\\ &\frac{1}{2\pi i}\int_o[-Z_u\xi+(\lambda-Z_w)\eta-(Z_q+U)\lambda\zeta]e^{\lambda t}d\lambda=P_2 e^{\mu t},\\ &\frac{1}{2\pi i}\int_o[-M_u\xi-M_w\eta+(k_B^2\lambda^2-M_q\lambda)\zeta]e^{\lambda t}d\lambda=P_3 e^{\mu t}.\end{aligned} \tag{20}$$

We next set

$$\begin{aligned}(\lambda-X_u)\xi-X_w\eta-(X_q\lambda+g)\zeta&=P_1/(\lambda-\mu),\\-Z_u\xi+(\lambda-Z_w)\eta-(Z_q+U)\lambda\zeta&=P_2/(\lambda-\mu),\\-M_u\xi-M_w\eta+(k_B{}^2\lambda^2-M_q\lambda)\zeta&=P_3/(\lambda-\mu),\end{aligned}\tag{21}$$

and solve for ξ, η, ζ, finding

$$\begin{aligned}\xi&=\frac{P_1\delta_{11}+P_2\delta_{21}+P_3\delta_{31}}{\Delta(\lambda-\mu)},\\\eta&=\frac{P_1\delta_{12}+P_2\delta_{22}+P_3\delta_{32}}{\Delta(\lambda-\mu)},\\\zeta&=\frac{P_1\delta_{13}+P_2\delta_{23}+P_3\delta_{33}}{\Delta(\lambda-\mu)},\end{aligned}\tag{22}$$

$$\Delta=34(\lambda^4+8.49\lambda^3+24.5\lambda^2+3.385\lambda+.917).$$

Bromwich shows that, if with these values of ξ, η, ζ we take the loop integrals (19) around a very large circle, the results for u, w, θ will be the solutions for the motion disturbed from rest at the position of equilibrium by the impressed forces P. As he points out, this integration is equivalent to the sum of the integrals around infinitesimal circles about $\lambda=\mu$ and about each of the roots λ of $\Delta=0$, that is, the integral is equal to the sum of the residues of $\xi e^{\lambda t}$, $\eta e^{\lambda t}$, $\zeta e^{\lambda t}$. There is no need to calculate any constants of integration. Moreover any of the quantities u, w, θ can be obtained without the others. The numerators in ξ, η, ζ are already calculated in (20 a, b, c) of page 59.

We have, for example, for a head gust u_1,

$$\xi=\frac{.128\lambda^3+1.16\lambda^2+3.385\lambda+.917}{(\lambda-\mu)(\lambda+4.18\pm2.43i)(\lambda+.0654\pm.187i)}u_1,\tag{23}$$

where the double sign stands for two factors, and $u_1=J(1-e^{-t})$, to take a particular case. The residues at each point are merely the values of the fraction when one of the factors, the one which vanishes at that point, is thrown out of the denominator. In the first case for $1=e^{0t}$ we have as residue of $\xi e^{\lambda t}=\xi$:

at $\lambda=\mu=0$,

$$-\frac{.917}{(+4.18\pm2.43i)(.0654\pm.187i)}=1;$$

at $\lambda=-4.18-2.43i$,

$$-\frac{.128\lambda^3+1.16\lambda^2+3.385\lambda+.917}{(-4.18+2.43i)(-4.86i)(-4.12-2.43i\pm.187i)};$$

at $\lambda=-4.18+2.43i$, the conjugate imaginary expression. And so on. To treat e^{-t} we should have:

at $\lambda=\mu=-1$,

$$-\frac{-.128+1.16-3.385+.917}{(3.18\pm2.43i)(.9346\pm.187i)};$$

and so on.

As the calculation with imaginaries involving squares, cubes, products, and quotients is by no means simple, it is clear that to get the solution for u will be reasonably hard work—much harder than to find the particular solutions which for the simple gust involved only real numbers. It may be admitted that to work any one gust the labor will probably be much less than by my method of determining formulas for the constants of integration in terms of the initial values of the particular integrals. But as far as I can see, Bromwich's method is of no particular advantage if we desire to calculate the effects of a large number of gusts $J\ (1-e^{-rt})$ of various degrees of sharpness both head-on, up, and rotary. When we came to calculate a periodic gust we found that we were involved in powers and products and quotients of complex numbers, and it is probable that the work we did in finding the particular integrals was comparable with that required for the present analysis.

SUMMARY.

In continuation of my previous work in gusts as affecting the Curtiss tractor *JN2*, I have discussed:

1. *Periodic longitudinal gusts.*—It was found that, even in the case of best resonance with the slow natural oscillation, the motion was not much different from that produced by a simple head-on gust until after a considerable time (over 14 seconds) had elapsed. The amplitude of the forced oscillation (in up and down motion) which ultimately became effective was about five times the amplitude of the gust. This was not regarded as serious because true periodicity can rarely be maintained in a head gust and because no pilot would wait to let its effect reach such a magnitude. Periodic up gusts and rotary gusts were considered as not likely to arise.

2. *General theory of resonance.*—It was shown that for aeroplane problems resonance meant different things for different problems. It was inferred that resonance was unlikely to be particularly serious because in all probability its effect would either be small or would take so long to become established that the pilot would check it.

3. *Infinitely sharp gusts.*—It was seen that the shock to a machine was mX_uJ and mZ_uJ for a head gust, and mX_wJ and mZ_wJ for an up gust. The serious case is mZ_wJ, the vertical shock in an up gust which was about $4J/g$ times the weight, more than twice that found for the sharpest gust previously treated. It would be still more serious in a machine where Z_w was greater than in the *JN2*. The moral: Keep Z_w small, clashes with Hunsaker's conclusion [1] that lateral stability is incompatible with high wing loading—but such an antithesis is common.[2] Reference was made to impulsive gusts.

4. *The effect of the propeller.*—The assumption that a constant power instead of a constant thrust was delivered did not very materially alter conditions of flight.

5. *Lateral gusts.*—The general equations were set up and integrated.

(*a*) Single side gusts were shown to produce modern side slipping, insignificant roll, and moderate yaw. It was seen that the yaw was into the relative wind so that the displacement of the machine in space was toward the gust despite the side slipping.

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[1] "Dynamical stability of aeroplanes," Washington, Smithsonian Misc. Collect., vol. 62, 1916, p. 77.
[2] "The production of a laterally stable aeroplane is attendant with many compromises," Hunsaker, p. 74

(*b*) Oscillatory side gusts were shown to be a common condition of flight, to produce moderate side slipping and yawing, but insignificant rolling. The path of the center of gravity proved to be sinusoidal, so far as the forced oscillation was concerned, and of amplitude about two or three times the amplitude of the gust.

(*c*) Yawing gusts were found to induce a good deal of side slipping, but did not appear to be serious. The roll was very small.

(*d*) Rolling gusts were seen to put the machine into a spiral dive, and thus to cause a real danger unless the motion were checked promptly by the pilot.

6. *Constrained machines.*—A device to keep the aeroplane on its course or to prevent rolling made the previously unstable machine stable. Such a device might be important to reduce the liability to the spiral dive in rolling gusts provided such gusts were common phenomena in flying weather.

7. *Other methods of treatment.*—The Bryan-Bordetsky method of initial motions and Bromwich's new method of finding the solution for a disturbed state without calculating the constants of integration were briefly compared with my system of analysis.